Isabelle's meta-logic

Basic constructs

Implication $\Longrightarrow$ (==>)

For separating premises and conclusion of theorems

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Equality $\equiv$ (`==`)

For definitions

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Universal quantifier $\bigwedge$ (!!)

For binding local variables

Basic constructs

Implication $\Longrightarrow$ (==>)

For separating premises and conclusion of theorems

Equality $\equiv$ (==)

For definitions

Universal quantifier $\bigwedge$ (!!)

For binding local variables

Do not use *inside* HOL formulae

Notation

$$\llbracket A_1; \ldots ; A_n \rrbracket \Longrightarrow B$$

abbreviates

$$A_1 \Longrightarrow \ldots \Longrightarrow A_n \Longrightarrow B$$

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$$[\![A_1; \ldots ; A_n]\!] \Longrightarrow B$$

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$$A_1 \Longrightarrow \ldots \Longrightarrow A_n \Longrightarrow B$$

$;$ $\approx$ “and”

The proof state

1. $\bigwedge x_1 \ldots x_p.\ [\![A_1; \ldots ; A_n]\!] \Longrightarrow B$

$x_1 \ldots x_p$	Local constants
$A_1 \ldots A_n$	Local assumptions
B	Actual (sub)goal

Type and function definition in Isabelle/HOL

Type definition in Isabelle/HOL

Introducing new types

Keywords:

- **typedecl**: pure declaration
- **types**: abbreviation
- **datatype**: recursive datatype

typedecl

typedecl $name$

Introduces new “opaque” type $name$ without definition

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Example:

typedecl *addr* — An abstract type of addresses

types

types $name = \tau$

Introduces an *abbreviation* $name$ for type τ

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Examples:

types
name = string
('a,'b)foo = 'a list $\times$ *'b list*

types

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Type abbreviations are expanded immediately after parsing
Not present in internal representation and Isabelle output

datatype

The example

datatype *'a list = Nil | Cons 'a 'a list*

Properties:

- Types: *Nil* :: *'a list*

  Cons :: *'a* $\Rightarrow$ *'a list* $\Rightarrow$ *'a list*
- Distinctness: *Nil* $\neq$ *Cons x xs*
- Injectivity: *(Cons x xs = Cons y ys) = (x = y* $\wedge$ *xs = ys)*

The general case

$$
\textbf{datatype } (\alpha_1, \ldots, \alpha_n)\tau \quad = \quad C_1\ \tau_{1,1} \ldots \tau_{1,n_1} \quad | \quad \ldots \quad | \quad C_k\ \tau_{k,1} \ldots \tau_{k,n_k}
$$

- *Types:* $C_i :: \tau_{i,1} \Rightarrow \cdots \Rightarrow \tau_{i,n_i} \Rightarrow (\alpha_1, \ldots, \alpha_n)\tau$
- *Distinctness:* $C_i\ \ldots \neq C_j\ \ldots$ if $i \neq j$
- *Injectivity:*
  $(C_i\ x_1 \ldots x_{n_i} = C_i\ y_1 \ldots y_{n_i}) = (x_1 = y_1 \wedge \ldots \wedge x_{n_i} = y_{n_i})$

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Distinctness and Injectivity are applied automatically
Induction must be applied explicitly

Function definition in Isabelle/HOL

Why nontermination can be harmful

How about $f\ x = f\ x + 1$?

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Subtract $f\ x$ on both sides.

$$\Longrightarrow 0 = 1$$

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! All functions in HOL must be total **!**

Function definition schemas in Isabelle/HOL

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  No problem

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- Well-founded recursion with **fun**
  Automatic termination proof
- Well-founded recursion with **function**
  User-supplied termination proof

definition

Definition (non-recursive) by example

definition $sq :: nat \Rightarrow nat$ **where** $sq\ n = n * n$

Definitions: pitfalls

definition *prime* $:: nat \Rightarrow bool$ **where**

$prime\ p = (1 < p \wedge (m\ dvd\ p \longrightarrow m = 1 \vee m = p))$

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prime p = (1 < p $\wedge$ *(m dvd p* $\longrightarrow$ *m = 1* $\vee$ *m = p))*

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! Every free variable on the rhs must occur on the lhs **!**

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$prime\ p = (1 < p \wedge (\forall m.\ m\ dvd\ p \longrightarrow m = 1 \vee m = p))$

Using definitions

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Unfolding the definition of *sq*:

apply*(unfold sq_def)*

primrec

The example

primrec

app Nil ys = ys

app (Cons x xs) ys = Cons x (app xs ys)

The general case

If τ is a datatype (with constructors $C_1, \ldots, C_k$) then
$f :: \cdots \Rightarrow \tau \Rightarrow \cdots \Rightarrow \tau'$ can be defined by *primitive recursion*:

$$f\ x_1 \ldots (C_1\ y_{1,1} \ldots y_{1,n_1}) \ldots x_p \quad = \quad r_1$$

$\vdots$

$$f\ x_1 \ldots (C_k\ y_{k,1} \ldots y_{k,n_k}) \ldots x_p \quad = \quad r_k$$

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$$f\ x_1 \ldots (C_k\ y_{k,1} \ldots y_{k,n_k}) \ldots x_p \quad = \quad r_k$$

The recursive calls in r_i must be *structurally smaller*,
i.e. of the form $f\ a_1 \ldots y_{i,j} \ldots a_p$

nat is a datatype

datatype *nat* = *0* | *Suc nat*

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datatype *nat = 0 | Suc nat*

Functions on *nat* definable by primrec!

primrec
f 0 = ...
f(Suc n) = ... f n ...

More predefined types and functions

Type option

datatype *'a option = None | Some 'a*

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Important application:

$$\ldots \Rightarrow \textit{'a option} \quad \approx \quad \text{partial function:}$$

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$$\textit{Some a} \quad \approx \quad \text{result } \textit{a}$$

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Example:

consts *lookup :: 'k* $\Rightarrow$ *('k* $\times$ *'v) list* $\Rightarrow$ *'v option*

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Example:

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primrec

lookup k [] = None

lookup k (x#xs) =

  (if fst x = k then Some(snd x) else lookup k xs)

case

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(case xs of [] ⇒ . . . | y#ys ⇒ ... y ... ys ...)

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Datatype values can be taken apart with *case* expressions:

(case xs of [] $\Rightarrow \ldots$ | *y#ys* $\Rightarrow$ *... y ... ys ...)*

Wildcards:

(case xs of [] $\Rightarrow$ *[]* | *y#_* $\Rightarrow$ *[y])*

Nested patterns:

(case xs of [0] $\Rightarrow$ *0* | *[Suc n]* $\Rightarrow$ *n* | *_* $\Rightarrow$ *2)*

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Complicated patterns mean complicated proofs!

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Nested patterns:

(case xs of [0] ⇒ 0 | [Suc n] ⇒ n | _ ⇒ 2)

Complicated patterns mean complicated proofs!

Needs *( )* in context

Case distinctions

If t :: τ and τ is a datatype

apply(*case_tac* t**)**

creates k subgoals

$$t = C_i \; x_1 \ldots x_p \Longrightarrow \ldots$$

one for each constructor C_i of type τ.

Demo: trees

fun

From primitive recursion to arbitrary pattern matching

Example: Fibonacchi

fun *fib* :: *nat* $\Rightarrow$ *nat* **where**

fib 0 = 0 |
fib (Suc 0) = 1 |
fib (Suc(Suc n)) = fib (n+1) + fib n

Example: Separation

fun $sep :: 'a \Rightarrow 'a\ list \Rightarrow 'a\ list$ **where**

sep a [] = [] |
sep a [x] = [x] |
sep a (x#y#zs) = x # a # sep a (y#zs)

Example: Ackermann

fun $ack :: nat \Rightarrow nat \Rightarrow nat$ **where**

$ack\ 0\ n = Suc\ n\ |$
$ack\ (Suc\ m)\ 0 = ack\ m\ (Suc\ 0)\ |$
$ack\ (Suc\ m)\ (Suc\ n) = ack\ m\ (ack\ (Suc\ m)\ n)$

Key features of fun

- Arbitrary pattern matching

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- Order of equations matters
- Termination must be provable
  by lexicographic combination of size measures

Size

- $size(n::nat) = n$

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- *size* counts number of (non-nullary) constructors

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Similar for tuples:

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Theorem If each component ordering terminates, then their *lexicographic product* terminates, too.

Ackermann terminates

$ack\ 0\ n = Suc\ n$

$ack\ (Suc\ m)\ 0 = ack\ m\ (Suc\ 0)$

$ack\ (Suc\ m)\ (Suc\ n) = ack\ m\ (ack\ (Suc\ m)\ n)$

Ackermann terminates

ack 0 n = Suc n

ack (Suc m) 0 = ack m (Suc 0)

ack (Suc m) (Suc n) = ack m (ack (Suc m) n)

because the arguments of each recursive call are lexicographically smaller than the arguments on the lhs.

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Note: order of arguments not important for Isabelle!

Computation Induction

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Induction follows course of (terminating!) computation

Computation Induction: Example

fun $div2 :: nat \Rightarrow nat$ **where**
$div2\ 0 = 0\ |$
$div2\ (Suc\ 0) = 0\ |$
$div2(Suc(Suc\ n)) = Suc(div2\ n)$

Computation Induction: Example

fun *div2 :: nat* $\Rightarrow$ *nat* **where**
div2 0 = 0 |
div2 (Suc 0) = 0 |
div2(Suc(Suc n)) = Suc(div2 n)

$\leadsto$ induction rule `div2.induct`:

$$\frac{P(0) \quad P(Suc\,0) \quad P(n) \Longrightarrow P(Suc(Suc\,n))}{P(m)}$$

Demo: fun