Exercise 6.1 Validity and satisfiability

Which of the following formulas is valid, which is satisfiable?

For each satisfiable formula give a model. For each formula that is satisfiable but not valid, give a counter-model.

a) \((\forall x.\exists y.P(x, y)) \Rightarrow (\exists y.\forall x.P(x, y))\)
b) \((\exists x.\forall y.P(x, y)) \Rightarrow (\forall y.\exists x.P(x, y))\)
c) \((\exists x.\forall y.P(x, y)) \Rightarrow (\forall x.\exists y.P(x, y))\)

Exercise 6.2 More validity, satisfiability and models

Prove or disprove, using the definitions:

a) For each sentence \(p\), either \(p\) is valid or \(\neg p\) is valid.
b) A sentence \(p\) is valid iff \(\neg p\) is unsatisfiable.

Exercise 6.3 Trivial model

Prove that a formula containing only \(\land, \lor, \forall, \exists, \Rightarrow\) and atomic formulas is always satisfiable.

Exercise 6.4 Extension with \(\exists!\)

a) Extend the definition of first-order formulas with an additional quantifier \(\exists!\), such that \(\exists! x.P(x)\) means that there exists a unique \(x\) such that \(P(x)\) holds. Modify the definition of the semantics.
b) Extend the OCaml implementations of the type \texttt{formula} and the function \texttt{holds} accordingly.
c) Is the following formula valid?

\[(\exists! x.\exists! y.P(x, y)) \Rightarrow (\exists! y.\exists! x.P(x, y))\]