Exercise 5.1  More Sequent Calculus

Prove that if the sequent $\Gamma \vdash \Delta$ is derivable in the Sequent Calculus, then so is $\Gamma[x \mapsto p] \vdash \Delta[x \mapsto p]$, where $\Gamma[x \mapsto p]$ denotes the set of formulas obtained by replacing all occurrences of $x$ in $\Gamma$ by $p$.

Give two different proofs:

a) one using a transformation of proof trees,

b) and one using soundness and completeness of SC.

Exercise 5.2

Use the compactness theorem to solve Exercise 2.20 from the book (p. 117).

Exercise 5.3  Subformula property

A rule has the subformula property iff all formulas in the premises are subformulas of the conclusion.

a) Check if all rules of the Sequent Calculus have the subformula property. Why is it important?

b) Find suitable sequent rules for the connectives $\overline{\lor}$ (“nand”) and $\otimes$ (“xor”). Make sure they have the subformula property.
Proof Rules of the Sequent Calculus:

Axioms
\[
\begin{align*}
&\Gamma, p \vdash \Delta, p \quad \text{Ax} \\
&\Gamma, \bot \vdash \Delta \quad \bot L \\
&\Gamma \vdash \Delta, \top \quad \top R
\end{align*}
\]

Rules
\[
\begin{align*}
&\Gamma \vdash \Delta, p \quad \Gamma \vdash \Delta, \neg p \\
&\Gamma, \neg p \vdash \Delta \quad \neg L \\
&\Gamma, p, q \vdash \Delta \quad \Gamma, p \vdash \Delta, q \\
&\Gamma, p \land q \vdash \Delta, \land L \\
&\Gamma \vdash \Delta, p \quad \Gamma \vdash \Delta, q \quad \Gamma \vdash \Delta, p \land q \\
&\Gamma \vdash \Delta, \land R \\
&\Gamma \vdash \Delta, p \quad \Gamma, q \vdash \Delta \\
&\Gamma, p \lor q \vdash \Delta, \lor L \\
&\Gamma \vdash \Delta, p, q \quad \Gamma \vdash \Delta, p \lor q \\
&\Gamma \vdash \Delta, \lor R \\
&\Gamma \vdash \Delta, p \quad \Gamma, q \vdash \Delta \\
&\Gamma, p \Rightarrow q \vdash \Delta, \Rightarrow L \\
&\Gamma \vdash \Delta, p \Rightarrow q \quad \Gamma, q \vdash \Delta, \Rightarrow R
\end{align*}
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